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The forcing geodetic global domination number of a graph

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Let G be a connected graph and S be a minimum geodetic global dominating set of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique minimum geodetic global dominating set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing geodetic global domination number of S, denoted by $f_{\overline{\gamma}g}(S)$, is the cardinality of a minimum forcing subset of S. The forcing geodetic global domination number of G, denoted by $f_{\overline{\gamma}g}(G)$, is $f_{\overline{\gamma}g}(G) = \min\{f_{\overline{\gamma}g}(S)\}$, where the minimum is taken over all minimum geodetic global dominating sets S in G. The forcing geodetic global domination number of some standard graphs are determined. Some of its general properties are studied. It is shown that for every pair of positive integers a and b with $0 \leq a \leq b$ and $b > a + 2$, there exists a connected graph G such that $f_{\overline{\gamma}g}(G) = a$ and $\overline{\gamma}(G) = b$. The geodetic global domination number of join of graphs is also studied. Connected graphs of order $n \geq 2$ with geodetic global domination number 2 are characterized. It is proved that, for a connected graph G with $\overline{\gamma}_g(G) = 2$. Then $0 \leq f_{\overline{\gamma}g}(G) \leq 1$ and characterized connected graphs for which the lower and the upper bounds are sharp.

Keywords: Forcing geodetic global domination number; geodetic global domination number; geodetic number; domination number; global domination number.

Mathematics Subject Classification 2020: 05C69, 05C12

1. Introduction

Let $G = (V, E)$ be a graph with a *vertex set* $V(G)$ and *edge set* $E(G) \subseteq V(G) \times V(G)$ (or simply V and E, respectively). Furthermore, we say that a graph G has order $n = |V(G)|$ and size $m = |E(G)|$. For basic graph theoretic terminology, we refer to [5]. A vertex v is adjacent to another vertex u if and only if there exists an edge $e = uv \in E(G)$. If $uv \in E(G)$, we say that u is a *neighbor* of v and denote by $N_G(v)$, the set of neighbors of v. The *degree* of a vertex $v \in V(G)$ is $\deg_G(v) = |N_G(v)|$. A vertex v is said to be a *universal vertex* if $\deg_G(v) = n - 1$. A vertex of degree 1 is called a *pendent vertex* or an *end vertex* of G. $G - \{e\}$ is the graph obtained from G by deleting an edge e from G. The maximum and minimum degree of G are denoted by $\triangle(G)$ and $\delta(G)$ respectively. The *subgraph induced* by a set S of vertices of a graph G is denoted by $\langle S \rangle$ with $V(\langle S \rangle) = S$ and $E(\langle S \rangle) = \{uv \in E(G) : u, v \in S\}$. A set S subset of $V(G)$ is called a *clique* if $\langle S \rangle$ is complete. The *clique number* of G is the number of vertices in a maximum clique and is denoted by $\omega(G)$. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighborhood is complete. Let G_1 and G_2 be two graphs. Then *join* $G_1 \vee G_2$ of two graphs G_1 and G_2 is the graph with $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

The length of a path is the number of its edges. Let u and v be vertices of a connected graph G. A shortest u-v path is also called a *u-v geodesic*. The (shortest path) distance is defined as the length of a u-v-geodesic in G and is denoted by $d_G(u, v)$ or $d(u, v)$. The maximum distance between two vertices of G is called *diameter* of G and is denoted by $d(G)$. Two vertices u and v of G are *antipodal* if $d(u, v) = \text{diam}G$ or $d(G)$. A vertex x is said to lie on a u-v geodesic P if x is a vertex of P including the vertices u and v. For two vertices u and v, the closed interval $I[u, v]$ consists of u and v together with all vertices lying in a u-v geodesic. If u and v are adjacent, then $I[u, v] = \{u, v\}$. For a set S of vertices, let $I[S] = \cup_{u,v \in S} I[u, v]$. Then certainly $S \subseteq I[S]$. A set $S \subseteq V(G)$ is called a *geodetic set* of G if $I[S] = V$. The *geodetic number* $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is a *g-set* of G. The geodetic number of a graph was studied in [1, 4, 8, 13, 15, 21]. Obviously $2 \leq g(G) \leq n$ for any connected graph G of order n. Among graphs on n vertices only complete graphs attain the upper bound, while the family of graphs that attain the lower bound is much richer. In [5], it is shown that $g(G) = 2$ if and only if there exist two vertices u and v with $d(u, v) = d(G)$ and every vertex of G lies on a geodesic between vertices u and v.

A set $D \subseteq V(G)$ is a *dominating* set of G if for every $v \in V(G) \backslash D$ is adjacent to some vertex in D. A dominating set D is said to be *minimal* if no proper subset of D is a dominating set of G. The minimum cardinality of a minimal dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. Any dominating set of cardinality $\gamma(G)$ is a *γ-set* of G. The domination number of a graph was studied in [9]. A *dominating clique set* is a dominating set that induces a complete subgraph. A subset $D \subseteq V(G)$ is called a *global dominating set* of G if D is a dominating set of both G and $\overline{G}$. A global dominating set D is said to be *minimal* if no proper subset of D is a global dominating set of G. The minimum cardinality of a minimal global dominating set of G is called the *global domination number* of G and is denoted by $\overline{\gamma}(G)$. Any global dominating set of cardinality $\overline{\gamma}(G)$ of G is a $\overline{\gamma}$-*set* of G. The global domination number of a graph was studied in [20, 23]. A set $S \subseteq V(G)$ is said to be a *geodetic global dominating set* of G if S is both a geodetic set and a global dominating set of G. The minimum cardinality of a geodetic global dominating set of G is the *geodetic global domination number* of G and is denoted by $\overline{\gamma}_g(G)$. A geodetic global dominating set of cardinality $\overline{\gamma}_g(G)$ is called a $\overline{\gamma}_g$-*set* of G. The geodetic global dominating number of a graph was studied in [18, 19]. Let G be a connected graph and D be a γ-set of G. A subset $T \subseteq D$ is called a forcing subset for D if D is the unique γ-set containing T. A forcing subset for D of minimum cardinality is a minimum forcing subset of D. The forcing domination number of D, denoted by $f_\gamma(D)$, is the cardinality of a minimum forcing subset of D. The forcing domination number of G denoted by $f_\gamma(G)$, is $f_\gamma(G) = \min\{f_\gamma(D)\}$ where the minimum is taken over γ-sets of D in G. By the similar manner, we can define the forcing global domination number $f_{\overline{\gamma}}(G)$ and the forcing geodetic number $f_g(G)$.

For the graph G given in Fig. 1, there are four γ-sets of G namely $D_1 = \{v_1, v_3\}$, $D_2 = \{v_2, v_3\}$, $D_3 = \{v_4, v_6\}$ and $D_4 = \{v_3, v_6\}$ so that $\gamma(G) = 2$. Since D_1 is the only γ-set containing v_1, it follows that $f_\gamma(D_1) = 1$. Also since D_2 is the only γ-set containing v_2, it follows that $f_\gamma(D_2) = 1$. Also since D_3 is the only γ-set containing v_4, it follows that $f_\gamma(D_3) = 1$. Since no singleton subsets of D_4 is a forcing subset of D_4, it follows that $f_\gamma(D_4) = 2$. Therefore $f_\gamma(G) = 1$. There are eleven $\overline{\gamma}$-sets namely $M_1 = \{v_1, v_2, v_3\}$, $M_2 = \{v_1, v_4, v_6\}$, $M_3 = \{v_2, v_3, v_5\}$, $M_4 = \{v_2, v_4, v_6\}$, $M_5 = \{v_3, v_4, v_6\}$, $M_6 = \{v_4, v_5, v_6\}$, $M_7 = \{v_1, v_3, v_4\}$, $M_8 = \{v_1, v_3, v_5\}$, $M_9 = \{v_1, v_4, v_5\}$, $M_{10} = \{v_2, v_3, v_4\}$, $M_{11} = \{v_3, v_5, v_6\}$. No singleton subsets of M_i is a forcing subset of M_i for all i $(1 \le i \le 6)$ and so $f_{\overline{\gamma}}(M_i) \ge 2$ for all i $(1 \le i \le 6)$. Since at least one two element subset of M_i $(1 \le i \le 6)$ is not a subset of M_j $(1 \le j \le 6; i \ne j)$, it follows that $f_{\overline{\gamma}}(M_i) = 2$ for all i $(1 \le i \le 6)$. Also since no singleton subsets or no two element subsets of M_i $(7 \le i \le 11)$ is a subset of M_i for all i $(7 \le i \le 11)$, it follows that $f_{\overline{\gamma}}(M_i) = 3$ for all i $(7 \le i \le 11)$. Therefore $f_{\overline{\gamma}}(G) = 2$. There are two g-sets of G namely, $S_1 = \{v_1, v_4, v_5\}$ and $S_2 = \{v_1, v_4, v_6\}$.

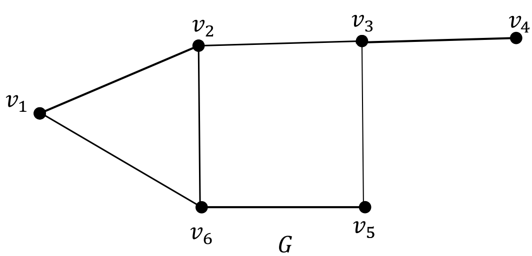

Fig. 1. A graph G with $f_\gamma(G) = 1$, $f_{\overline{\gamma}}(G) = 2$, $f_g(G) = 1$, $\gamma(G) = 2$, $\overline{\gamma}(G) = 3$ and $g(G) = 3$.

Therefore $g(G) = 3$ and $f_g(G) = 1$. The forcing concepts in graphs was introduced in [6] for minimum dominating sets. Then the forcing concepts was further studied by many authors in several parameters arising in graphs [2, 3, 6, 7, 10–12, 14, 16, 17, 22].

In Sec. 2, we introduce the concepts of forcing geodetic global dominating number of a graph and determined this for some standard graphs. It is shown that for every pair of positive integers a and b with $0 \leq a \leq b$ and $b > a + 2$, there exists a connected graph G such that $f_{\overline{\gamma}g}(G) = a$ and $\overline{\gamma}(G) = b$. In Sec. 3, we studied the concepts of the geodetic global domination number of join of graphs. In Sec. 4, we characterized connected graphs of order $n \geq 2$ with geodetic global domination number 2. It is proved that, for a connected graph G with $\overline{\gamma}_g(G) = 2$, $0 \leq f_{\overline{\gamma}g}(G) \leq 1$ and characterized connected graphs for which the lower and the upper bounds are sharp.

A social network theory is concerned about the study of relationships between the members of a group. A social network clique is a dominating clique which is a group of representatives of the network who can communicate themselves directly. A status in a network is a subset S of members of the group such that any two of them have the same relationship outside S in the network. A group of people is said to be structurally equivalent if any two of them have same relationship between people in the social network. These sets can be determined using the properties of dominating sets. Concepts related to geodesics also appear in the social sciences. When members of a social group are represented by vertices, and particular relationship between members by edges of a graph, then the geodetic number can be considered as the smallest number p such that every member of a group is contained in a minimal chain of relationships between chosen p members. By applying the geodetic global domination concepts, there must be effectiveness in the relationships between chosen p members. The following theorems are used in sequel.

Theorem 1.1 ([19]). *If v is either an extreme vertex or a universal vertex of a connected graph G, then v belongs to every geodetic global dominating set of G.*

Theorem 1.2 ([19]). *Let G be a connected graph of order $n \geq 2$. Then $\overline{\gamma}_g(G) = 2$ if and only if either $G = K_2$ or there exists a geodetic set $S = \{x, y\}$ such that $d(x, y) = 3$.*

Theorem 1.3 ([19]). *If G is either a complete graph K_n $(n \geq 2)$ (or) a star $K_{1,n-1}$ $(n \geq 3)$, then $\overline{\gamma}_g(G) = n$.*

Theorem 1.4 ([5]). *For a connected graph G, $g(G) = 2$ if and only if there exist peripheral vertices u and v such that every edge of G is on a diametral path joining u and v.*

2. The Forcing Geodetic Global Domination Number of a Graph

Definition 2.1. Let G be a connected graph and S be a minimum geodetic global dominating set of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique minimum geodetic global dominating set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing geodetic global domination number of S, denoted by $f_{\overline{\gamma}g}(S)$, is the cardinality of a minimum forcing subset of S. *The forcing geodetic global domination number* of G, denoted by $f_{\overline{\gamma}g}(G)$, is $f_{\overline{\gamma}g}(G) = \min\{f_{\overline{\gamma}g}(S)\}$, where the minimum is taken over all minimum geodetic global dominating sets S in G.

Example 2.2. For the graph G given in Fig. 2, $S_1 = \{v_1, v_3, v_5\}$, $S_2 = \{v_1, v_4, v_6\}$ and $S_3 = \{v_1, v_4, v_5\}$ are the only three minimum geodetic global dominating sets of G such that $f_{\overline{\gamma}g}(S_1) = f_{\overline{\gamma}g}(S_2) = 1$ and $f_{\overline{\gamma}g}(S_3) = 2$ so that $f_{\overline{\gamma}g}(G) = 1$.

Definition 2.3. A vertex v is said to be a geodetic global dominating vertex of G if v belongs to every minimum geodetic global dominating set of G. If G has a unique minimum geodetic global dominating set S, then every vertex of S is a geodetic global dominating vertex of G.

Example 2.4. For the graph G given in Fig. 3, $S_1 = \{v_1, v_4, v_7, v_9\}$ and $S_2 = \{v_1, v_4, v_7, v_8\}$ are the only two $\overline{\gamma}g$-sets of G. Since v_1, v_4 and $v_7 \in S_1 \wedge S_2$, it follows that $\{v_1, v_4, v_7\}$ is the set of all geodetic global dominating vertices of G.

Note 2.5. Each extreme vertex and each universal vertex of G are geodetic global. dominating vertices of G. In fact there are geodetic global dominating vertices which are neither extreme vertices nor universal vertices of G. For the graph G given in Fig. 3, v_4 is a geodetic global dominating vertex of G which is neither an extreme vertex nor a universal vertex of G.

The next theorem follows immediately from the definition of the geodetic global domination number and the forcing geodetic global domination number of a connected graph G.

Theorem 2.6. *For every connected graph G, $0 \le f_{\overline{\gamma}g}(G) \le \overline{\gamma}_g(G)$.*

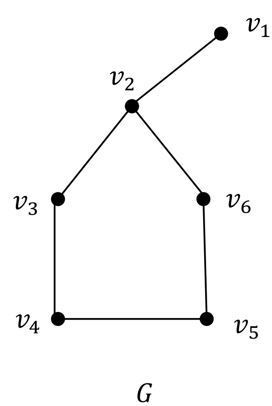

Fig. 2. A graph G with $f_{\overline{\gamma}_g}(G) = 1$ and $\overline{\gamma}_g(G) = 3$.

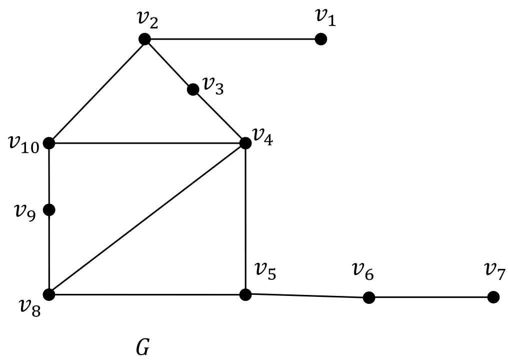

Fig. 3. A graph G with $\{v_1, v_4, v_7\}$ are the set of all geodetic global domination vertices.

In the following, we characterize graphs G for which the bounds in Theorem 2.6 attained and also graphs for which $f_{\overline{\gamma}g}(G) = 1$. The proofs of the following theorems are straightforward. So we omit the proofs.

Theorem 2.7. (a) $f_{\overline{\gamma}g}(G) = 0$ *if and only if* G *has a unique minimum geodetic global dominating set.*

(b) $f_{\overline{\gamma}g}(G) = 1$ *if and only if* G *has at least two minimum geodetic global dominating. sets, one of which is a unique minimum geodetic global dominating set containing one of its elements.*

(c) $f_{\overline{\gamma}g}(G) = \overline{\gamma}_g(G)$ *if and only if no minimum geodetic global dominating set of* G *is the unique minimum geodetic global dominating set containing any of its proper subsets.*

Theorem 2.8. *Let* G *be a connected graph and let* $\Im$ *be the set of relative complements of the minimum forcing subsets in their respective minimum geodetic global dominating sets in* G. *Then* $\cap_{F\in\Im} F$ *is the set of geodetic global dominating vertices of* G.

Corollary 2.9. *Let* G *be a connected graph and* S *a minimum geodetic global dominating set of* G. *Then no geodetic global dominating vertex of* G *belongs to any minimum forcing set of* S.

Theorem 2.10. *Let* G *be a connected graph and* X *be the set of all geodetic global dominating vertices of* G. *Then* $f_{\overline{\gamma}g}(G) \leq \overline{\gamma}_g(G) - |X|$.

In the following, we determine the forcing geodetic global domination number of some standard graphs.

Theorem 2.11. *If* G *is either a complete graph* K_n $(n \geq 2)$ *or a star* $K_{1,n-1}$ $(n \geq 3)$, *then* $f_{\overline{\gamma}g}(G) = 0$.

Proof. Since $S = V(G)$ is the unique $\overline{\gamma}_g$-set of G, the result follows from Theorem 2.7(a). □

Theorem 2.12. *For a double star G, $f_{\overline{\gamma}g}(G) = 0$.*

Proof. Let Z be the set of all end vertices of G. Then Z is the unique $\overline{\gamma}_g$-set of G so that $f_{\overline{\gamma}g}(G) = 0$. □

Theorem 2.13. *For the cycle $G = C_n$ $(n \geq 4)$,*

$$f_{\overline{\gamma}g}(G) = \begin{cases} 3, & \textit{if } n = 4, \\ 1, & \textit{if } n \equiv 0 \pmod 3, \\ 2, & \textit{otherwise.} \end{cases}$$

Proof. Let C_n be $v_1, v_2, \ldots, v_n, v_1$.

Case 1. Let $n = 4$. Then $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_2, v_3, v_4\}$, $S_3 = \{v_3, v_4, v_1\}$ and $S_4 = \{v_4, v_1, v_2\}$ are the only four minimum geodetic global dominating sets of G such that $f_{\overline{\gamma}g}(S_1) = f_{\overline{\gamma}g}(S_2) = f_{\overline{\gamma}g}(S_3) = f_{\overline{\gamma}g}(S_4) = 3$ so that $f_{\overline{\gamma}g}(G) = 3$.

Case 2. Let $n = 5$. Then $S_1 = \{v_1, v_3, v_4\}$, $S_2 = \{v_1, v_2, v_4\}$, $S_3 = \{v_1, v_3, v_5\}$, $S_4 = \{v_2, v_4, v_5\}$ and $S_5 = \{v_2, v_3, v_5\}$ are the only five minimum geodetic global dominating sets of G such that $f_{\overline{\gamma}g}(S_i) = 2$ for $1 \leq i \leq 5$ so that $f_{\overline{\gamma}g}(G) = 2$.

Case 3. Let $n \equiv 0 \pmod 3$. Let $n = 3k$, $k \geq 2$. Then $S = \{v_1, v_4, v_7, \ldots, v_{3k-2}\}$ is the unique $\overline{\gamma}_g$-*set* of G containing $\{v_1\}$ so that $f_{\overline{\gamma}g}(G) = 1$.

Case 4. Let $n \equiv 1 \pmod 3$. Let $n = 3k+1$, $k \geq 2$. Let S be any $\overline{\gamma}_g$-set of G. Then it is easily verified that any singleton subset of S is a subset of another $\overline{\gamma}_g$-set of G and so $f_{\overline{\gamma}g}(G) \geq 2$. Now, $S_1 = \{v_1, v_4, v_7, \ldots, v_{3k+1}\}$ is the unique $\overline{\gamma}_g$-set of G containing $\{v_1, v_{3k+1}\}$ so that $f_{\overline{\gamma}g}(G) = 2$.

Case 5. Let $n + 1 \equiv 0 \pmod 3$. Let $n = 3k - 1$, $k \geq 3$. Let S be any $\overline{\gamma}_g$-set of G. Then it is easily verified that any singleton subset of S is a subset of another $\overline{\gamma}_g$-set of G and so $f_{\overline{\gamma}g}(G) \geq 2$. Now, $S_1 = \{v_1, v_4, v_7, \ldots, v_{3k-2}\}$ is the unique $\overline{\gamma}_g$-set of G containing $\{v_1, v_{3k-2}\}$ so that $f_{\overline{\gamma}g}(G) = 2$. □

Theorem 2.14. *For the path $G = P_n$ $(n \geq 4)$,*

$$f_{\overline{\gamma}g}(G) = \begin{cases} 0, & \textit{if } n-1 \equiv 0 \pmod 3 \quad \textit{and} \quad n = 4, \\ 1, & \textit{if } n \equiv 0 \pmod 3 \quad \textit{and} \quad n = 5, \\ 2, & \textit{if } n+1 \equiv 0 \pmod 3 \quad \textit{and} \quad n \geq 8. \end{cases}$$

Proof. Let P_n be $v_1, v_2, v_3, \ldots, v_n$.

Case 1. Let $n = 4$. Then $S_1 = \{v_1, v_4\}$ is the unique $\overline{\gamma}_g$-set of G such that $f_{\overline{\gamma}g}(G) = 0$.

Case 2. Let $n - 1 \equiv 0 \pmod 3$. Let $n = 3k + 1$, $k \geq 2$. Then $S = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k+1}\}$ is the unique $\overline{\gamma}_g$-set of G so that $f_{\overline{\gamma}g}(G) = 0$.

Case 3. Let $n \equiv 0 \pmod 3$. Let $n = 3k$, $k = 2$. Then $S_1 = \{v_1, v_3, v_6\}$ and $S_2 = \{v_1, v_4, v_6\}$ are the only two $\overline{\gamma}_g$-sets of G such that $f_{\overline{\gamma}g}(S_1) = f_{\overline{\gamma}g}(S_2) = 1$ so that $f_{\overline{\gamma}g}(G) = 1$. Next assume that $k \geq 3$. Then $S = \{v_1, v_3, v_6, v_9, \ldots, v_{3k}\}$ is the unique $\overline{\gamma}_g$-set of G containing $\{v_3\}$ so that $f_{\overline{\gamma}g}(G) = 1$.

Case 4. Let $n + 1 \equiv 0 \pmod 3$. Let $n = 3k - 1$, First assume that $k = 2$. Then $S_1 = \{v_1, v_2, v_5\}$, $S_2 = \{v_1, v_3, v_5\}$ and $S_3 = \{v_1, v_4, v_5\}$ are the only three $\overline{\gamma}_g$-set of G such that $f_{\overline{\gamma}g}(S_1) = f_{\overline{\gamma}g}(S_2) = f_{\overline{\gamma}g}(S_3) = 1$ so that $f_{\overline{\gamma}g}(G) = 1$. Next assume that $k \geq 3$. Then $S = \{v_1, v_4, v_7, \ldots, v_{3k-5}, v_{3k-2}, v_{3k-1}\}$ is the unique $\overline{\gamma}_g$-set of G containing $\{v_{3k-5}, v_{3k-2}\}$ so that $f_{\overline{\gamma}g}(G) = 2$. □

Theorem 2.15. *For the fan graph $G = K_1 \vee P_{n-1} (n \geq 5)$,*

$$f_{\overline{\gamma}g}(G) = \begin{cases} 0, & \text{if } n-1 \text{ is odd,} \\ 1, & \text{if } n-1 \text{ is even.} \end{cases}$$

Proof. Let $V(K_1) = \{x\}$ and $V(P_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$. We have the following cases.

Case 1. $n - 1$ is odd. Let $n - 1 = 2k + 1$. $(k \geq 2)$. Then $S = \{x, v_1, v_3, \ldots, v_{2k+1}\}$ is the unique $\overline{\gamma}_g$-set of G so that $f_{\overline{\gamma}g}(G) = 0$.

Case 2. $n-1$ is even. Let $n-1 = 2k$. $(k \geq 3)$. Then $S = \{x, v_1, v_2, v_4, \ldots, v_{2k-2}, v_{2k}\}$ is the unique $\overline{\gamma}_g$-set of G containing $\{v_{2k-2}\}$ so that $f_{\overline{\gamma}g}(G) = 1$. □

Theorem 2.16. *For the wheel graph $G = K_1 \vee C_{n-1}$ $(n \geq 5)$,*

$$f_{\overline{\gamma}g}(G) = \begin{cases} \dfrac{n-2}{2}, & \text{if } n-1 \text{ is odd,} \\ 1, & \text{if } n-1 \text{ is even.} \end{cases}$$

Proof. Let $V(K_1) = \{x\}$ and $V(C_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$. We have the following cases.

Case 1. $n - 1$ is odd. Let $n - 1 = 2k + 1$. $(k \geq 2)$. Then $S = \{x, v_1, v_3, v_5, v_7, \ldots, v_{2k+1}\}$ is a $\overline{\gamma}_g$-set of G. Since $d(u, v) \leq 2$ for all $u, v \in V(G)$, it follows that there is no forcing subset T of S with $|T| \leq k-1$. Now S is the unique $\overline{\gamma}_g$-set of G containing $\{v_1, v_5, v_7, \ldots, v_{2k+1}\}$ so that $f_{\overline{\gamma}g}(G) = k$. Since this true for all $\overline{\gamma}_g$-sets S in G, it follows that $f_{\overline{\gamma}g}(G) = k = \frac{n-2}{2}$.

Case 2. $n - 1$ is even. Let $n - 1 = 2k$. $(k \geq 2)$. Then $S_1 = \{x, v_1, v_3, \ldots, v_{2k-1}\}$ and $S_2 = \{x, v_2, v_4, \ldots, v_{2k}\}$ are two $\overline{\gamma}_g$-sets of G such that $f_{\overline{\gamma}g}(S_1) = f_{\overline{\gamma}g}(S_2) = 1$ so that $f_{\overline{\gamma}g}(G) = 1$. □

Theorem 2.17. *For the graph $G = K_n - \{e\}$, $f_{\overline{\gamma}g}(G) = 0$.*

Proof. Since every vertex of G is either an extreme vertex or a universal vertex, $S = V(G)$ is the unique $\overline{\gamma}_g$-set of G, it follows from Theorems 1.1 and 2.7 (a) that $f_{\overline{\gamma}g}(G) = 0$. □

Theorem 2.18. *Let $F_n = K_1 \vee P_{n-1}$, $(n \geq 7)$ be the fan graph. Let $V(K_1) = \{x\}$ and $G = F_n - \{e\}$, where $e \in E(P_{n-1})$ and G_1, G_2 be the components of $G - x$. Then*

$$f_{\overline{\gamma}g}(G) = \begin{cases} 0, & \text{if } |V(G_1)| \quad \text{and} \quad |V(G_2)| \text{ are odd} \\ 1, & \text{if either } |V(G_1)| \quad \text{or} \quad |V(G_2)| \text{ is odd} \\ 2, & \text{if } |V(G_1)| \quad \text{and} \quad |V(G_2)| \text{ are even.} \end{cases}$$

Proof. Let S be a $\overline{\gamma}_g$-set of G. Then by Theorem 1.1, $x \in S$. Let $V(G_1) = \{v_1, v_2, \ldots, v_l\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_r\}$, where $l + r \geq 6$,

Case 1. $|V(G_1)|$ and $|V(G_2)|$ are odd. Let $l = 2k + 1$ and $r = 2s + 1$, where $k \geq 1$ and $s \geq 1$. Then $S = \{x\} \cup \{v_1, v_3, ...v_{2k+1}\} \cup \{u_1, u_2, \ldots, u_{2s+1}\}$ is the unique $\overline{\gamma}_g$-set of G so that $f_{\overline{\gamma}g}(G) = 0$.

Case 2. $|V(G_1)|$ is odd and $|V(G_2)|$ is even. Let $l = 2k+1$ and $r = 2s$, where $k \geq 1$ and $s \geq 2$. Then $Z = \{x\} \cup \{v_1, v_3, \ldots, v_{2k+1}\} \cup \{u_1, u_{2s}\}$ is the set of all geodetic global dominating vertices of G. Since $|V(G_1)|$ is odd and $|V(G_2)|$ is even, $\overline{\gamma}_g$-set of G is not unique and so $f_{\overline{\gamma}g}(G) \geq 1$. Now $S = Z \cup \{u_3, u_5, \ldots u_{2s-1}\}$ is the unique $\overline{\gamma}_g$-set of G containing $\{u_{2s-1}\}$. Hence it follows that $f_{\overline{\gamma}g}(G) = 1$.

Case 3. $|V(G_1)|$ and $|V(G_2)|$ are even. Let $l = 2k$ and $r = 2s$, where $k \geq 2$ and $s \geq 2$. Then $Z = \{x\} \cup \{v_1, v_{2k}\} \cup \{u_1, u_{2k}\}$ is the set of all geodetic global dominating vertices of G. Since $|V(G_1)|$ and $|V(G_2)|$ are even, no singleton subsets of any $\overline{\gamma}_g$-set S of G is a forcing subset of S and so $f_{\overline{\gamma}g}(G) \geq 2$. Now $S = Z \cup \{v_3, v_5, \ldots, v_{2k-1}\} \cup \{u_3, u_5, \ldots, u_{2s-1}\}$ is the unique $\overline{\gamma}_g$-set of G containing $\{v_{2k-1}, u_{2s-1}\}$. Hence it follows that $f_{\overline{\gamma}g}(G) = 2$. □

In view of Theorem 2.6, we have the following realization result.

Theorem 2.19. *For every pair of positive integers a and b with $0 \leq a < b$ and $b > a + 2$, there exists a connected graph G such that $f_{\overline{\gamma}g}(G) = a$ and $\overline{\gamma}_g(G) = b$.*

Proof. For $a = 0$, $b = 2$, let $G = K_2$. Then $S = V(G)$ is the unique $\overline{\gamma}_g$-set of G so that $f_{\overline{\gamma}g}(G) = 0$ and $\overline{\gamma}_g(G) = 2$. Let $a = 1$, $b \geq 3$. Let C_5 be $v_1, v_2, v_3, v_4, v_5, v_1$. Let G be the graph obtained from C_5 by adding new vertices $z_1, z_2, \ldots, z_{b-2}$ and introducing the edge $v_1 z_i$ $(1 \leq i \leq b-2)$. The graph is shown in Fig. 4.

Let $Z = \{z_1, z_2, \ldots, z_{b-2}\}$ be the set of all end vertices of G. Then by Theorem 1.1, Z is a subset of every geodetic global dominating set of G. Now $S_1 = Z \cup \{v_3, v_4\}$, $S_2 = Z \cup \{v_2, v_4\}$, $S_3 = Z \cup \{v_3, v_5\}$ are the only three $\overline{\gamma}_g$-sets of G such that $f_{\overline{\gamma}g}(S_1) = 2, f_{\overline{\gamma}g}(S_2) = f_{\overline{\gamma}g}(S_3) = 1$ so that $f_{\overline{\gamma}g}(G) = 1$ and $\overline{\gamma}_g(G) = b$.

Let $2 \leq a < b$ and $b > a+2$. Let $P : x, y, z$ be a path of order 3 and $Q_i : u_i, v_i, w_i$ $(1 \leq i \leq a)$ be a copy path of order 3. Let H be a graph obtained from P and Q_i $(1 \leq i \leq a)$ by joining y with each u_i $(1 \leq i \leq a)$ and z with each w_i $(1 \leq i \leq a)$. Let G be a graph obtained from H by adding new vertices $w, z_1, z_2, \ldots, z_{b-a-2}$

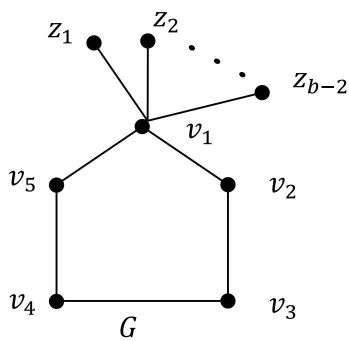

Fig. 4. A graph G with $f_{\overline{\gamma}_g}(G) = 1$ and $\overline{\gamma}_g(G) = b$.

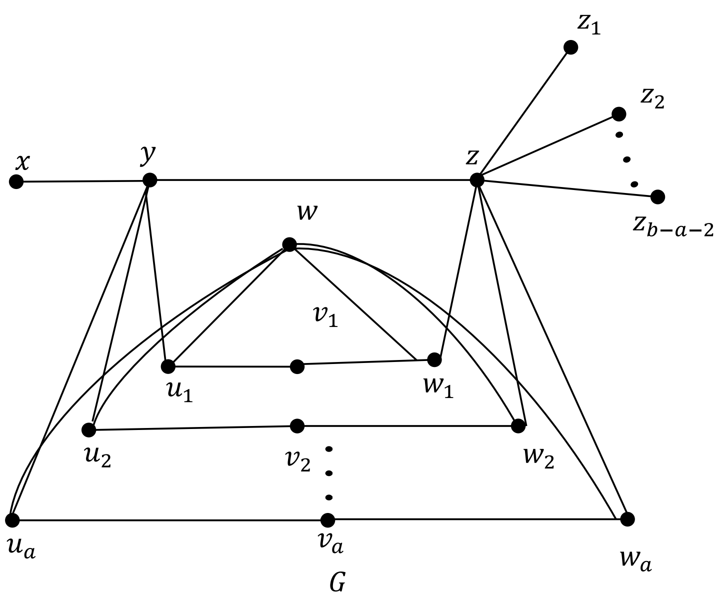

Fig. 5. A graph G with $f_{\overline{\gamma}_g}(G) = a$ and $\overline{\gamma}_g(G) = b$.

and joining w with each u_i $(1 \leq i \leq a)$ and each w_i $(1 \leq i \leq a)$; z with each z_i $(1 \leq i \leq b-a-2)$. The graph G is shown in Fig. 5.

First we show that $\overline{\gamma}_g(G) = b$. Let $Z = \{x, w, z_1, z_2, \ldots, z_{b-a-2}\}$ be the set of all geodetic global dominating vertices of G and $H_i = \{u_i, v_i, w_i\}$ $(1 \leq i \leq a)$. By Theorem 1.1, Z is a subset of every geodetic global dominating set of G. Let D be a minimum geodetic dominating set of G. We prove that D contains at least one element in each H_i $(1 \leq i \leq a)$. On the contrary, suppose that D contains no elements of H_i, for every i, $(1 \leq i \leq a)$. Then v_i $(1 \leq i \leq a)$ is not dominated by any element of D, which is a contradiction. Therefore every minimum geodetic dominating set of G contains at least one vertex from each H_i $(1 \leq i \leq a)$ and so

$\overline{\gamma}_g(G) \geq 2+b-a-2+a = b$. Now $S = Z \cup \{u_1, u_2, \ldots, u_a\}$ is a geodetic dominating set of G so that $\overline{\gamma}_g(G) = b$.

Next we show that $f_{\overline{\gamma}g}(G) = a$. By Theorem 2.10, $f_{\overline{\gamma}g}(G) \leq \overline{\gamma}_g - |Z| = a$. Now since $\overline{\gamma}_g(G) = b$ and every minimum geodetic global dominating set of G contains Z, and $\overline{\gamma}_g$-set S of G is of the form $S = Z \cup \{t_1, t_2, \ldots, t_a\}$, where $t_i \in H_i$ $(1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then there exists a vertex t_j $(1 \leq j \leq a)$ such that $t_j \notin T$. Let g_j be a vertex of H_j distinct from t_j. Then $S_1 = S - \{t_j\} \cup \{g_j\}$ is a $\overline{\gamma}_g$-set of G properly containing T. Therefore T is not a forcing subset of S. This is true for all $\overline{\gamma}_g$-sets of G. Hence it follows that $f_{\overline{\gamma}g}(G) = a$. □

3. The Forcing Geodetic Global Domination Number of Join of Graphs

Theorem 3.1. *Let G_1 and G_2 be two complete graphs of order n_1 and n_2 respectively. Then $f_{\overline{\gamma}g}(G_1 \vee G_2) = 0$.*

Proof. Since $G = K_{n_1}$ and $G = K_{n_2}$, we have $G_1 \vee G_2 = K_{n_1+n_2}$. Then by Theorem 2.11, $f_{\overline{\gamma}g}(G_1 \vee G_2) = 0$. □

Theorem 3.2. *Let G_1 and G_2 be two non-empty graphs of order n_1 and n_2 respectively. If $\overline{\gamma}_g(G_1 \vee G_2) = 2$, then $f_{\overline{\gamma}g}(G_1 \vee G_2) = 0$.*

Proof. Let $\overline{\gamma}_g(G_1 \vee G_2) = 2$. Then by Theorem 1.2, $G_1 \vee G_2$ is either (i) K_2 or (ii) there exists a geodetic set $S = \{x, y\}$ in $G_1 \vee G_2$ such that $d_{G_1 \vee G_2}(x, y) = 3$. Since $d(G_1 \vee G_2) = 2$, (ii) is not possible. Therefore $G = K_2$. Hence by Theorem 2.11, $f_{\overline{\gamma}g}(G_1 \vee G_2) = 0$. □

Theorem 3.3. *Let G_1 and G_2 be graphs of order n_1 and n_2, respectively. If $\overline{\gamma}_g(G_1 \vee G_2) = 3$. Then $f_{\overline{\gamma}g}(G_1 \vee G_2) \leq 1$ or $f_{\overline{\gamma}g}(G_1 \vee G_2) = 3$.*

Proof. Let $\overline{\gamma}g(G_1 \vee G_2) = 3$. Then by Theorem 2.6, $0 \leq f_{\overline{\gamma}g}(G_1 \vee G_2) \leq 3$. Let $S = \{x, y, z\}$ be a $\overline{\gamma}_g$-set of $G_1 \vee G_2$. Since $d(G_1 \vee G_2) = 2$ and $I_{G_1 \vee G_2}[S] = V(G_1 \vee G_2)$, S contains at least two vertices from G_1 (or) at least two vertices from G_2. Let us assume that $x, y \in V(G_1)$. If $xy \in E(G_1)$, then $G_1 \vee G_2 = K_3$. Hence by Theorem 2.11, $f_{\overline{\gamma}g}(G_1 \vee G_2) = 0$. If $xy \notin E(G_1)$, then we prove that $V(G_1) = \{x, y\}$. On the contrary suppose that $V(G_1) \neq \{x, y\}$. Then there exists $w \in V(G_1)$ such that $w \neq x$ and $w \neq y$. Since every vertex in G_2 is adjacent to x, y and w, $w \notin I_{G_1 \vee G_2}[S]$, then it follows that S is not a geodetic global dominating set of $G_1 \vee G_2$, which is a contradiction. Therefore $V(G_1) = \{x, y\}$ and so $G_1 = \overline{K}_2$. Since $zx, zy \in E(\overline{G_1 \vee G_2})$ and $zu \in E(\overline{G_1 \vee G_2})$ for every $u \in V(G_1 \vee G_2)$ and $u \neq x, y$, $\delta(G_2) = 0$. Hence it follows that $\deg_{G_2}(z) = 0$. If $G_2 = K_1$, then $G_1 \vee G_2 = P_3$. By Theorem 2.11, $f_{\overline{\gamma}g}(G_1 \vee G_2) = 0$. If $G_2 = \overline{K_2}$, then $G_1 \vee G_2 = C_4$. By Theorem 2.13, $f_{\overline{\gamma}g}(G_1 \vee G_2) = 3$. If G_2 is neither K_1 nor $\overline{K_2}$, then G_2 contains at least three vertices.

We prove that $f_{\overline{\gamma}g}(G_1 \vee G_2) \neq 2$. On the contrary suppose that $f_{\overline{\gamma}g}(G_1 \vee G_2) = 2$. Then there exists a $\overline{\gamma}_g$-set S_1 of $G_1 \vee G_2$ such that S_1 contains at least two vertices of G_2. Let $S = \{x, z, z_1\}$, where $z, z_1 \in G_2$. Let $z_2 \in V(G_2)$ such that $z_2 \neq z, z_1$. Since $\deg_{G_2}(z) = 0$, $d_{G_1 \vee G_2}(z, z_1) = 2$ and $xz, xz_1, xz_2 \in E(G_1 \vee G_2)$, then it follows that $z_2 \notin I[S_1]$. Hence S_1 is not a geodetic global dominating set of $G_1 \vee G_2$, which is a contradiction. Therefore $f_{\overline{\gamma}g}(G_1 \vee G_2) \neq 2$. □

Theorem 3.4. *Let G be a connected graph of order $n_1 \geq 3$. Then $f_{\overline{\gamma}g}(G \vee K_{n_2}) = f_{\overline{\gamma}g}(G)$.*

Proof. Let S be a $\overline{\gamma}_g$-set of G and T be a forcing subset of S. Since $d_{G \vee K_2}(x, y) = 1$ for all $x, y \in V(K_2)$ and $xu \in E(G \vee K_2)$ for all $x \in V(K_{n_2})$ and $u \in V(G)$, it follows that $V(K_2)$ is a subset of every $\overline{\gamma}_g$-set of $G \vee K_{n_2}$. Therefore $S \cup V(K_{n_2})$ is a $\overline{\gamma}_g$-set of $G \vee K_{n_2}$. We prove that T is a forcing subset of $S \cup V(K_{n_2})$. On the contrary, suppose that T is not a forcing subset of $S \cup V(K_{n_2})$. Then there exists a $\overline{\gamma}_g$-set $S_1 \cup V(K_{n_2})$ of $G \vee K_{n_2}$ such that $T \subset S_1 \cup V(K_{n_2})$, where S_1 is a $\overline{\gamma}_g$-set of G. Since $V(K_{n_2})$ is a subset of every $\overline{\gamma}_g$-set of $G \vee K_{n_2}$. It follows that T is a subset of S_1, which is a contradiction. Therefore T is a forcing subset of $S \cup V(K_{n_2})$. Since this is a true for all forcing subset T of S, we have $f_{\overline{\gamma}g}(G) \geq f_{\overline{\gamma}g}(G \vee K_{n_2})$.

Let M be a $\overline{\gamma}_g$-set of $G \vee K_{n_2}$. Then $M = M_1 \cup V(K_{n_2})$, where M_1 is a $\overline{\gamma}_g$-set of G. Let K be a forcing subset of M. Since $V(K_{n_2})$ is a subset of every $\overline{\gamma}_g$-set of $G \vee K_{n_2}$, it follows that K is a forcing subset of M_1. Since this is true for all forcing subset K of M, we have $f_{\overline{\gamma}g}(G \vee K_{n_2}) \geq f_{\overline{\gamma}g}(G)$. Therefore $f_{\overline{\gamma}g}(G \vee K_{n_2}) = f_{\overline{\gamma}g}(G)$. □

4. Graphs with Geodetic Global Domination Number 2

In [15], connected graphs of order $n \geq 2$ with geodetic global domination number n are characterized. In this section, we characterize connected graphs of order $n \geq 2$ with $\overline{\gamma}_g(G) = 2$. For this purpose we introduce the graphs H and the following family $\Im$ of graphs.

Let $V(K_4 - \{e\}) = \{v_1, v_2, v_3, v_4\}$ where $e = v_2v_4$. Let H be the graph obtained from $K_4 - \{e\}$ by introducing the end edge v_2v_5. Let $V(\overline{K}_2) = \{x, y\}$ and $V(\overline{K}_a) = \{v_1, v_2, \ldots, v_a\} (a \geq 2)$. Let $\Im_1$ be the graph obtained from $\overline{K}_2$ and $\overline{K}_a$ by adding a vertex z and introducing the edge xv_i $(1 \leq i \leq a)$, zv_i $(1 \leq i \leq a)$ and yz.

Let $V(\overline{K}_2) = \{x, y\}$ and $P_i : u_i, v_i$ $(1 \leq i \leq a)$ $(a \geq 1)$ be a copy of path on two vertices. Let $\Im_2$ be the graph obtained from $\overline{K}_2$ and P_i $(1 \leq i \leq a)$ by introducing the edges xu_i $(1 \leq i \leq a)$ and yv_i $(1 \leq i \leq a)$. Let $\Im_3$ be the graph obtained from $\Im_2$ by joining at least one u_i $(1 \leq i \leq a)$ with at least one u_j $(1 \leq j \leq a)$ and $i \neq j$. Let $\Im_4$ be the graph obtained from $\Im_2$ by joining at least one v_i $(1 \leq i \leq a)$ with at least one v_j $(1 \leq j \leq a)$, $i \neq j$. Let $\Im_5$ be the graph obtained from $\Im_2$ by joining at least one u_i $(1 \leq i \leq a)$ to at least one v_j $(1 \leq j \leq a)$, $i \neq j$. Let $\Im_6$ be the graph obtained from $\Im_5$ by joining at least one v_l $(1 \leq l \leq a)$ with at least one u_m $(1 \leq m \leq a)$, $l \neq m$. Let $\Im_7$ be the graph obtained from $\Im_3$ by joining

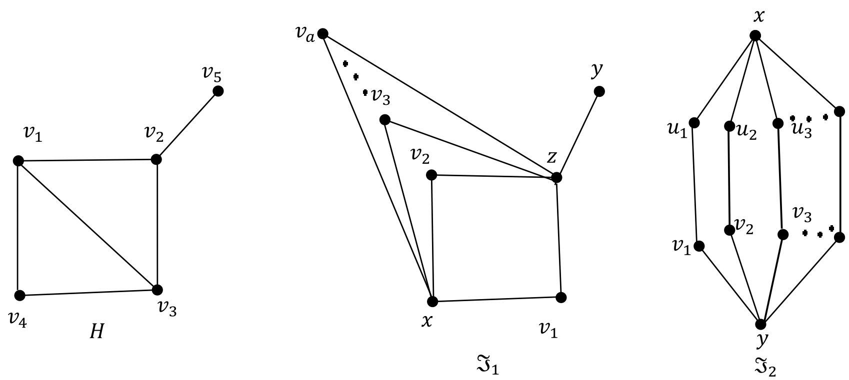

Fig. 6. Family $\mathcal{T}$ of graphs.

at least one u_l $(1 \leq l \leq a)$ with at least one v_m $(1 \leq m \leq a)$, $l \neq m$. Let $\Im_8$ be the graph obtained from $\Im_4$ by joining at least one v_l $(1 \leq l \leq a)$ with at least one u_m $(1 \leq m \leq a)$, $l \neq m$. Let $\Im_9$ be the graph obtained from $\Im_3$ by joining at least one v_l $(1 \leq l \leq a)$ with at least one v_m $(1 \leq m \leq a)$, $l \neq m$. Let $\Im_{10}$ be the graph obtained from $\Im_9$ by joining at least one u_r $(1 \leq r \leq a)$ with at least one v_s $(1 \leq s \leq a)$, $r \neq s$. Let $\Im_{11}$ be the graph obtained from $\Im_{10}$ by joining at least one u_p $(1 \leq p \leq a)$ with at least one v_q $(1 \leq q \leq a)$, $p \neq q$.

Theorem 4.1. *Let G be a connected graph of order $n \geq 2$. Then $\overline{\gamma}_g(G) = 2$ if and only if G is either K_2 or P_4 or C_6 or $G \in \{H, \Im_1, \Im_2, \Im_3, \Im_4, \Im_5, \Im_6, \Im_7, \Im_8, \Im_9, \Im_{10}, \Im_{11}\}$.*

Proof. Let $\overline{\gamma}_g(G) = 2$. Then $g(G) = 2$. If $n = 2$, then $G = K_2$, which satisfies the requirements of this theorem. If $n = 3$, then G is either K_3 or P_3. Then by Theorem 1.3, $\overline{\gamma}_g(G) = 3$, which is a contradiction. So let $n \geq 4$. By Theorem 1.2, $d = 3$. If $n = 4$ then $G = P_4$, which satisfies the requirements of this theorem. Hence $n \geq 5$. If G is a tree, then G contains at least three end vertices and so $\overline{\gamma}_g(G) \geq 3$, which is a contradiction. Therefore G is not a tree. If $G = C_n$ $(n \geq 5)$, then by Theorem 1.2, $G = C_6$, which satisfies the requirements of this theorem. So let $G \neq C_n$ $(n \geq 5)$. Let $S = \{x, y\}$ be a $\overline{\gamma}_g$-set of G. Since $g(G) = 2$, by Theorem 1.4, x and y are antipodal vertices of G. Since G is not a tree, either x or y is an end vertex of G, or x and y are not end vertices of G. If either x or y is an end vertex of G, then $G \in \{H, \Im_1\}$, which satisfies the requirement of this theorem. If neither x nor y is an end vertex of G, then $G \in \{\Im_2, \Im_3, \Im_4, \Im_5, \Im_6, \Im_7, \Im_8, \Im_9, \Im_{10}, \Im_{11}\}$, which satisfies the requirements of this theorem. The converse is clear. □

Theorem 4.2. *If G is a connected graph with $\overline{\gamma}_g(G) = 2$, then $0 \leq f_{\overline{\gamma}g}(G) \leq 1$.*

Proof. By Theorem 2.6, $0 \leq f_{\overline{\gamma}g}(G) \leq 2$. Let $S = \{u, v\}$ be any minimum geodetic global dominating set of G. Then by Theorem 1.4, u and v are antipodal vertices

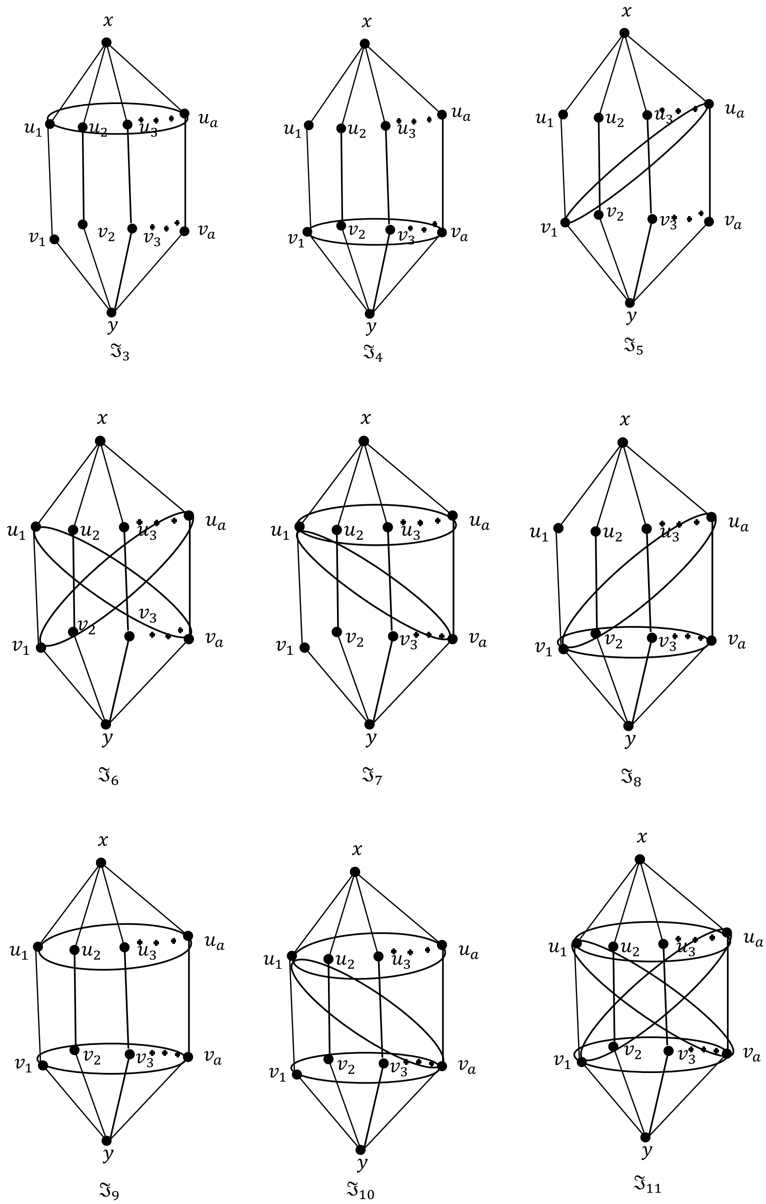

Fig. 7. Family $\mathcal{T}$ of graphs continued.

of G. Suppose that $f_{\overline{\gamma}g}(G) = 2$. Then $f_{\overline{\gamma}g}(S) = 2$ for all $\overline{\gamma}_g$-set S of G. Hence it follows that S is not a unique $\overline{\gamma}_g$-set containing u. Then there exists $x \neq u$ such that $S_1 = \{u, x\}$ is also a $\overline{\gamma}_g$-set of G. Then by Theorem 1.2, u and x are two antipodal vertices of G. Hence v is an internal vertex of some $u - x$ geodesic in G. Therefore, $d(u, v) < d(u, x)$, which is a contradiction. □

Theorem 4.3. *Let G be a connected graph with $\overline{\gamma}_g(G) = 2$. Then G contains more than one $\overline{\gamma}_g$-sets if and only if the degree of each element in any $\overline{\gamma}_g$-set of G is two.*

Proof. Let $S = \{x, y\}$ be a $\overline{\gamma}_g$-set of G. Let us assume that G contains more than one $\overline{\gamma}_g$-sets. Let $S_1 = \{u, v\}$ be another $\overline{\gamma}_g$-set of G. We prove that $\deg(x) = \deg(y) = \deg(u) = \deg(v) = 2$. On the contrary, suppose that degree of at least one element in the $\overline{\gamma}_g$-set is not two. Without loss of generality, let us assume that $\deg(x) \neq 2$. If $\deg(x) = 1$, then $\overline{\gamma}_g$-set is unique, which is a contradiction. Therefore $\deg(x) \geq 3$. Since the edges incident with x are not end vertices, there is a x-y geodesic, say P which avoids u or v. Let w be an internal vertex of P. Then w may not lie on a u-v geodesic, which is a contradiction to $S_1 = \{u, v\}$ is a $\overline{\gamma}_g$-set of G. Conversely let degree of each element in any $\overline{\gamma}_g$-sets is 2. We prove that G contains more than one $\overline{\gamma}_g$-sets. Let $M = \{x, y\}$ be a $\overline{\gamma}_g$-set of G. Let P be a x-y geodesic in G. Since $\deg(x) = \deg(y) = 2$, there is another x-y geodesic, say P_1 in G. By Theorem 1.2, $d(x, y) = 3$. Then there exists internal vertices u in P and v in P_1 such that $d(u, v) = 3$. Hence it follows that u and v are antipodal vertices of G. Since $\overline{\gamma}_g(G) = 2$, every vertex of G lies in u-v geodesic. Hence it follows that $S_1 = \{u, v\}$ is another $\overline{\gamma}_g$-set of G. Therefore G contains more than one $\overline{\gamma}_g$-sets. □

Theorem 4.4. *Let G be a connected graph with $\overline{\gamma}_g(G) = 2$. Then $f_{\overline{\gamma}g}(G) = 1$ if and only if G is either C_6 or the graph G given in Fig. 8.*

Proof. Let $f_{\overline{\gamma}g}(G) = 1$. Then by the Theorem 2.7(b), $\overline{\gamma}_g$-set of G is not unique. Let $S = \{x, y\}$ and $S_1 = \{u, v\}$ be two $\overline{\gamma}_g$-sets of G. Then by Theorem 4.3, $\deg(x) = \deg(y) = \deg(u) = \deg(v) = 2$. Hence it follows from Theorem 4.1, that G is either C_6 or the graph G given in Fig. 8. The converse is clear. □

Theorem 4.5. *Let G be a connected graph with $\overline{\gamma}_g(G) = 2$. Then $f_{\overline{\gamma}g}(G) = 0$ if and only if $G \in \{H, \Im_1, \Im_2(a \geq 3), \Im_3, \Im_4, \Im_5(a \geq 3), \Im_6, \Im_7, \Im_8, \Im_9, \Im_{10}, \Im_{11}\}$.*

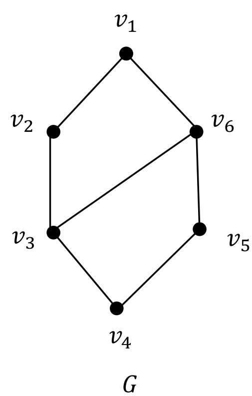

Fig. 8. A graph G with $f_{\overline{\gamma}g}(G) = 1$ and $\overline{r}_g(G) = 2$.

Proof. This follows from Theorems 4.1–4.4. □

Acknowledgments

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